

Topological Properties of Strong Solutions for the 3D Navier-Stokes Equations

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Abstract

In this paper we give a criterion for the existence of global strong solutions for the 3D Navier-Stokes system for any regular initial data.

Keywords: 3D Navier-Stokes System, Strong Solution

1. Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$ and $0 < T < +\infty$. We consider the incompressible Navier-Stokes equations

$$\begin{cases} y_t + (y \cdot \nabla)y = \nu \Delta y - \nabla p + f & \text{in } Q = \Omega \times (0, T), \\ \operatorname{div} y = 0 & \text{in } Q, \\ y = 0 & \text{on } \partial\Omega \times (0, T), \quad y(x, 0) = y_0(x) & \text{in } \Omega, \end{cases} \quad (1)$$

where $\nu > 0$ is a constant. We define the usual function spaces

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\},$$

$$H = \text{closure of } \mathcal{V} \text{ in } (L^2(\Omega))^3, \quad V = \{u \in (H_0^1(\Omega))^3 : \operatorname{div} u = 0\}.$$

We denote by V^* the dual space of V . The spaces H and V are separable Hilbert spaces and $V \subset H \subset V^*$ with dense and compact embedding when H is identified with its dual H^* . Let (\cdot, \cdot) , $\|\cdot\|_H$ and $((\cdot, \cdot))$, $\|\cdot\|_V$ be the inner product and the norm in H and V , respectively, and let $\langle \cdot, \cdot \rangle$ be the pairing between V and V^* . For $u, v, w \in V$, the equality

$$b(u, v, w) = \int_{\Omega} \sum_{i,j=1}^3 u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

defines a trilinear continuous form on V with $b(u, v, v) = 0$ when $u \in V$ and $v \in (H_0^1(\Omega))^3$. For $u, v \in V$, let $B(u, v)$ be the element of V^* defined by $\langle B(u, v), w \rangle = b(u, v, w)$ for all $w \in V$.

We say that the function y is a *weak solution* of Pr. (1) on $[0, T]$, if $y \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $\frac{dy}{dt} \in L^1(0, T; V^*)$, if

$$\frac{d}{dt}(y, v) + \nu((y, v)) + b(y, y, v) = \langle f, v \rangle \quad \text{for all } v \in V, \quad (2)$$

in the sense of distributions on $(0, T)$, and if y satisfies the energy inequality

$$V(y)(t) \leq V(y)(s) \quad \text{for all } t \in [s, T], \quad (3)$$

for a.e. $s \in (0, T)$ and for $s = 0$, where

$$V(y)(t) := \frac{1}{2} \|y(t)\|_H^2 + \nu \int_0^t \|y(\tau)\|_V^2 d\tau - \int_0^t \langle f(\tau), y(\tau) \rangle d\tau. \quad (4)$$

This class of solutions is called Leray–Hopf or physical one. If $f \in L^2(0, T; V^*)$, and if y satisfies (2), then $y \in C([0, T]; H_w)$, $\frac{dy}{dt} \in L^{\frac{4}{3}}(0, T; V^*)$, where H_w denotes the space H endowed with the weak topology. In particular, the initial condition $y(0) = y_0$ makes sense for any $y_0 \in H$.

Let $A : V \rightarrow V^*$ be the linear operator associated to the bilinear form $((u, v)) = \langle Au, v \rangle$. Then A is an isomorphism from $D(A)$ onto H with $D(A) = (H^2(\Omega))^3 \cap V$. We recall that the embedding $D(A) \subset V$ is dense and continuous. Moreover, we assume $\|Au\|_H$ as the norm on $D(A)$, which is equivalent to the one induced by $(H^2(\Omega))^3$. The Problem (1) can be rewritten as

$$\begin{cases} \frac{dy}{dt} + \nu Ay + B(y, y) = f \text{ in } V^*, \\ y(0) = y_0, \end{cases} \quad (5)$$

where the first equation we understand in the sense of distributions on $(0, T)$. Now we write

$$\mathcal{D}(y_0, f) = \{y : y \text{ is a weak solution of Pr. (1) on } [0, T]\}.$$

It is well known (cf. [1]) that if $f \in L^2(0, T; V^*)$, and if $y_0 \in H$, then $\mathcal{D}(y_0, f)$ is not empty.

A weak solution y of Pr. (1) on $[0, T]$ is called a *strong* one, if it additionally belongs to Serrin's class $L^8(0, T; (L^4(\Omega))^3)$. We note that any strong solution y of Pr. (1) on $[0, T]$ belongs to $C([0, T]; V) \cap L^2(0, T; D(A))$ and $\frac{dy}{dt} \in L^2(0, T; H)$ (cf. [2, Theorem 1.8.1, p. 296] and references therein).

For any $f \in L^\infty(0, T; H)$ and $y_0 \in V$ it is well known the only local existence of strong solutions for the 3D Navier-Stokes equations (cf. [2, 1, 3] and references therein). Here we provide a criterion for existence of strong solutions for Pr. (1) on $[0, T]$ for any initial data $y_0 \in V$ and $0 < T < +\infty$.

2. Topological Properties of Strong Solutions

The main result of this note has the following form.

Theorem 2.1. *Let $f \in L^2(0, T; H)$ and $y_0 \in V$. Then either for any $\lambda \in [0, 1]$ there is an $y_\lambda \in C([0, T]; V) \cap L^2(0, T; D(A))$ such that $y_\lambda \in \mathcal{D}(\lambda y_0, \lambda f)$, or the set*

$$\{y \in C([0, T]; V) \cap L^2(0, T; D(A)) : y \in \mathcal{D}(\lambda y_0, \lambda f), \lambda \in (0, 1)\} \quad (6)$$

is unbounded in $L^8(0, T; (L^4(\Omega))^3)$.

In the proof of Theorem 2.1 we use an auxiliary statement connected with continuity property of strong solutions on parameters of Pr. (1) in Serrin's class $L^8(0, T; (L^4(\Omega))^3)$.

Theorem 2.2. *Let $f \in L^2(0, T; H)$ and $y_0 \in V$. If y is a strong solution for Pr. (1) on $[0, T]$, then there exist $L, \delta > 0$ such that for any $z_0 \in V$ and $g \in L^2(0, T; H)$, satisfying the inequality*

$$\|z_0 - y_0\|_V^2 + \|g - f\|_{L^2(0, T; H)}^2 < \delta, \quad (7)$$

the set $\mathcal{D}(z_0, g)$ is one-point set $\{z\}$ which belongs to $C([0, T]; V) \cap L^2(0, T; D(A))$, and

$$\|z - y\|_{C([0, T]; V)}^2 + \frac{\nu}{2} \|z - y\|_{D(A)}^2 \leq L \left(\|z_0 - y_0\|_V^2 + \|g - f\|_{L^2(0, T; H)}^2 \right). \quad (8)$$

Remark 2.3. *We note that from Theorem 2.2 with $z_0 \in V$ and $g \in L^2(0, T; H)$ with $\|z_0\|_V^2 + \|g\|_{L^2(0, T; H)}^2$ sufficiently small, Problem (1) has only one global strong solution.*

Remark 2.4. *Theorem 2.2 provides that, if for any $\lambda \in [0, 1]$ there is an $y_\lambda \in L^8(0, T; (L^4(\Omega))^3)$ such that $y_\lambda \in \mathcal{D}(\lambda y_0, \lambda f)$, then the set*

$$\{y \in C([0, T]; V) \cap L^2(0, T; D(A)) : y \in \mathcal{D}(\lambda y_0, \lambda f), \lambda \in (0, 1)\}$$

is bounded in $L^8(0, T; (L^4(\Omega))^3)$.

If Ω is a C^∞ -domain and if $f \in C_0^\infty(\overline{(0, T) \times \Omega})^3$, then any strong solution y of Pr. (1) on $[0, T]$ belongs to $C^\infty((0, T] \times \Omega)^3$ and $p \in C^\infty((0, T] \times \Omega)$ (cf. [2, Theorem 1.8.2, p. 300] and references therein). This fact directly provides the next corollary of Theorems 2.1 and 2.2.

Corollary 2.5. *Let Ω be a C^∞ -domain, $f \in C_0^\infty(\overline{(0, T) \times \Omega})^3$. Then either for any $y_0 \in V$ there is a strong solution of Pr. (1) on $[0, T]$, or the set*

$$\{y \in C^\infty((0, T] \times \Omega)^3 : y \in \mathcal{D}(\lambda y_0, \lambda f), \lambda \in (0, 1)\}$$

is unbounded in $L^8(0, T; (L^4(\Omega))^3)$ for some $y_0 \in C_0^\infty(\Omega)^3$.

3. Proof of Theorem 2.2

Let $f \in L^2(0, T; H)$, $y_0 \in V$, and $y \in C([0, T]; V) \cap L^2(0, T; D(A))$ be a strong solution of Pr. (1) on $[0, T]$. Due to [4], [1, Chapter 3] the set $\mathcal{D}(y_0, f) = \{y\}$. Let us now fix $z_0 \in V$ and $g \in L^2(0, T; H)$ satisfying (7) with

$$\delta = \min \left\{ 1, \frac{\nu}{4} \right\} e^{-2TC}, \quad C = \max \left\{ \frac{27c^4}{2\nu^3}; \frac{7^8 c^8}{2^{12} \nu^7} \right\} \left(\|y\|_{C([0, T]; V)}^4 + 1 \right)^2, \quad (9)$$

$c > 0$ is a constant from the inequalities (cf. [2, 1])

$$|b(u, v, w)| \leq c \|u\|_V \|v\|_V^{\frac{1}{2}} \|v\|_{D(A)}^{\frac{1}{2}} \|w\|_H \quad \forall u \in V, v \in D(A), w \in H; \quad (10)$$

$$|b(u, v, w)| \leq c \|u\|_{D(A)}^{\frac{3}{4}} \|u\|_V^{\frac{1}{4}} \|v\|_V \|w\|_H \quad \forall u \in D(A), v \in V, w \in H. \quad (11)$$

The auxiliary problem

$$\begin{cases} \frac{d\eta}{dt} + \nu A\eta + B(\eta, \eta) + B(y, \eta) + B(\eta, y) = g - f \text{ in } V^*, \\ \eta(0) = z_0 - y_0, \end{cases} \quad (12)$$

has a strong solution $\eta \in C([0, T]; V) \cap L^2(0, T; D(A))$ with $\frac{d\eta}{dt} \in L^2(0, T; H)$, i.e.

$$\frac{d}{dt}(\eta, v) + \nu((\eta, v)) + b(\eta, \eta, v) + b(y, \eta, v) + b(\eta, y, v) = \langle g - f, v \rangle \quad \text{for all } v \in V,$$

in the sense of distributions on $(0, T)$. In fact, let $\{w_j\}_{j \geq 1} \subset D(A)$ be a special basis (cf. [5, p. 56]), i.e. $Aw_j = \lambda_j w_j$, $j = 1, 2, \dots$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$, $\lambda_j \rightarrow +\infty$, $j \rightarrow +\infty$. We consider Galerkin approximations $\eta_m : [0, T] \rightarrow \text{span}\{w_j\}_{j=1}^m$ for solutions of Pr. (12) satisfying

$$\frac{d}{dt}(\eta_m, w_j) + \nu((\eta_m, w_j)) + b(\eta_m, \eta_m, w_j) + b(y, \eta_m, w_j) + b(\eta_m, y, w_j) = \langle g - f, w_j \rangle,$$

with $(\eta_m(0), w_j) = (z_0 - y_0, w_j)$, $j = \overline{1, m}$. Due to (10), (11) and Young's inequality we get

$$\begin{aligned}
2\langle g - f, A\eta_m \rangle &\leq 2\|g - f\|_H \|\eta_m\|_{D(A)} \leq \frac{\nu}{4} \|\eta_m\|_{D(A)}^2 + \frac{4}{\nu} \|f - g\|_H^2; \\
-2b(\eta_m, \eta_m, A\eta_m) &\leq 2c \|\eta_m\|_V^{\frac{3}{2}} \|\eta_m\|_{D(A)}^{\frac{3}{2}} \leq \frac{\nu}{2} \|\eta_m\|_{D(A)}^2 + \frac{27c^4}{2\nu^3} \|\eta_m\|_V^6; \\
-2b(y, \eta_m, A\eta_m) &\leq 2c \|y\|_V \|\eta_m\|_V^{\frac{1}{2}} \|\eta_m\|_{D(A)}^{\frac{3}{2}} \leq \frac{\nu}{2} \|\eta_m\|_{D(A)}^2 + \frac{27c^4}{2\nu^3} \|y\|_{C([0,T];V)}^4 \|\eta_m\|_V^2; \\
-2b(\eta_m, y, A\eta_m) &\leq 2c \|\eta_m\|_{D(A)}^{\frac{7}{4}} \|\eta_m\|_V^{\frac{1}{4}} \|y\|_V \leq \frac{\nu}{2} \|\eta_m\|_{D(A)}^2 + \frac{7^8 c^8}{2^{12} \nu^7} \|y\|_{C([0,T];V)}^8 \|\eta_m\|_V^2.
\end{aligned}$$

Thus,

$$\frac{d}{dt} \|\eta_m\|_V^2 + \frac{\nu}{4} \|\eta_m\|_{D(A)}^2 \leq C(\|\eta_m\|_V^2 + \|\eta_m\|_V^6) + \frac{4}{\nu} \|g - f\|_H^2,$$

where $C > 0$ is a constant from (9). Hence, the absolutely continuous function $\varphi = \min\{\|\eta_m\|_V^2, 1\}$ satisfies the inequality $\frac{d}{dt} \varphi \leq 2C\varphi + \frac{4}{\nu} \|g - f\|_H^2$, and therefore $\varphi \leq L(\|z_0 - y_0\|_V^2 + \|g - f\|_{L^2(0,T;H)}^2) < 1$ on $[0, T]$, where $L = \delta^{-1}$. Thus, $\{\eta_m\}_{m \geq 1}$ is bounded in $L^\infty(0, T; V) \cap L^2(0, T; D(A))$ and $\{\frac{d}{dt} \eta_m\}_{m \geq 1}$ is bounded in $L^2(0, T; H)$. In a standard way we get that the limit function η of η_m , $m \rightarrow +\infty$, is a strong solution of Pr. (12) on $[0, T]$. Due to [4], [1, Chapter 3] the set $\mathcal{D}(z_0, g)$ is one-point $z = y + \eta \in L^8(0, T; (L^4(\Omega))^3)$. So, z is strong solution of Pr. (1) on $[0, T]$ satisfying (8).

The theorem is proved.

4. Proof of Theorem 2.1

We provide the proof of Theorem 2.1. Let $f \in L^2(0, T; H)$ and $y_0 \in V$. We consider the 3D controlled Navier-Stokes system (cf. [7, 6])

$$\begin{cases} \frac{dy}{dt} + \nu Ay + B(z, y) = f, \\ y(0) = y_0, \end{cases} \quad (13)$$

where $z \in L^8(0, T; (L^4(\Omega))^3)$.

By using standard Galerkin approximations (see [1]) it is easy to show that for any $z \in L^8(0, T; (L^4(\Omega))^3)$ there exists a unique weak solution $y \in L^\infty(0, T; H) \cap L^2(0, T; V)$ of Pr. (13) on $[0, T]$, that is,

$$\frac{d}{dt} (y, v) + \nu((y, v)) + b(z, y, v) = \langle f, v \rangle, \text{ for all } v \in V, \quad (14)$$

in the sense of distributions on $(0, T)$. Moreover, by the inequality

$$|b(u, v, Av)| \leq c_1 \|u\|_{(L^4(\Omega))^3} \|v\|_V^{\frac{1}{4}} \|v\|_{D(A)}^{\frac{7}{4}} \leq \frac{\nu}{2} \|v\|_{D(A)}^2 + c_2 \|u\|_{(L^4(\Omega))^3}^8 \|v\|_V^2, \quad (15)$$

for all $u \in (L^4(\Omega))^3$ and $v \in D(A)$, where $c_1, c_2 > 0$ are some constants that do not depend on u, v (cf. [1]), we find that $y \in C([0, T]; V) \cap L^2(0, T; D(A))$ and $B(z, y) \in L^2(0, T; H)$, so $\frac{dy}{dt} \in L^2(0, T; H)$ as well. We add that, for any $z \in L^8(0, T; (L^4(\Omega))^3)$ and corresponding weak solution $y \in C([0, T]; V) \cap L^2(0, T; D(A))$ of (13) on $[0, T]$, by using Gronwall inequality, we obtain

$$\begin{aligned} \|y(t)\|_V^2 &\leq \|y_0\|_V^2 e^{2c_2 \int_0^t \|z(t)\|_{(L^4(\Omega))^3}^8 dt}, \quad \forall t \in [0, T]; \\ \nu \int_0^T \|y(t)\|_{D(A)}^2 dt &\leq \|y_0\|_V^2 \left[1 + 2c_2 e^{2c_2 \int_0^T \|z(t)\|_{(L^4(\Omega))^3}^8 dt} \|z\|_{L^8(0, T; (L^4(\Omega))^3)}^8 \right]. \end{aligned} \quad (16)$$

Let us consider the operator $F : L^8(0, T; (L^4(\Omega))^3) \rightarrow L^8(0, T; (L^4(\Omega))^3)$, where $F(z) \in C([0, T]; V) \cap L^2(0, T; D(A))$ is the unique weak solution of (13) on $[0, T]$ corresponded to $z \in L^8(0, T; (L^4(\Omega))^3)$.

Let us check that F is a compact transformation of Banach space $L^8(0, T; (L^4(\Omega))^3)$ into itself (cf. [8]). In fact, if $\{z_n\}_{n \geq 1}$ is a bounded sequence in $L^8(0, T; (L^4(\Omega))^3)$, then, due to (15) and (16), the respective weak solutions $y_n, n = 1, 2, \dots$, of Pr. (13) on $[0, T]$ are uniformly bounded in $C([0, T]; V) \cap L^2(0, T; D(A))$ and their time derivatives $\frac{dy_n}{dt}, n = 1, 2, \dots$, are uniformly bounded in $L^2(0, T; H)$. So, $\{F(z_n)\}_{n \geq 1}$ is a precompact set in $L^8(0, T; (L^4(\Omega))^3)$. In a standard way we deduce that $F : L^8(0, T; (L^4(\Omega))^3) \rightarrow L^8(0, T; (L^4(\Omega))^3)$ is continuous mapping.

Since F is a compact transformation of $L^8(0, T; (L^4(\Omega))^3)$ into itself, Schaefer's Theorem (cf. [8, p. 133] and references therein) and Theorem 2.2 provide the statement of Theorem 2.1. We note that Theorem 2.2 implies that the set $\{z \in L^8(0, T; (L^4(\Omega))^3) : z = \lambda F(z), \lambda \in (0, 1)\}$ is bounded in $L^8(0, T; (L^4(\Omega))^3)$ iff the set defined in (6) is bounded in $L^8(0, T; (L^4(\Omega))^3)$.

The theorem is proved.

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